

“Magneto-elastic” waves in an anisotropic magnetised plasma

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The linear waves that propagate in a two fluid magnetised plasma allowing for a non-gyrotropic perturbed ion pressure tensor are investigated. For perpendicular propagation and perturbed fluid velocity a low frequency (magnetosonic) and a high frequency (ion Bernstein) branch are identified and discussed. For both branches a comparison is made with the results of a truncated Vlasov treatment. For the low frequency branch we show that a consistent expansion procedure allows us to recover the correct expression of the Finite Larmor Radius corrections to the magnetosonic dispersion relation.

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I. INTRODUCTION

The inclusion of the full pressure tensor evolution in the fluid equations of a magnetised plasma allows for a self-consistent description within a fluid framework of important effects that would otherwise require a kinetic treatment. This approach was used, e.g. to study collisionless magnetic reconnection driven by electron pressure anisotropy [1–5] or to investigate the dispersion relation of Geodesic Acoustic Modes [6, 7], within the so-called Grad hydrodynamic equations framework recovering in this way “kinetic” effects at a much lower computational cost. In addition, a macroscopic point of view can help to evidenciate dynamical processes that may be difficult to identify within the more complex kinetic description. In particular it was recently shown that a spatial inhomogeneity in the fluid velocity shear can make the pressure tensor non-gyrotropic [8]: in the case of a spatially localised sheared flow, the formation of a non-gyrotropic ion pressure tensor was shown to be mediated by the propagation in the direction orthogonal to the magnetic field of high frequency waves that can be interpreted as the fluid counterpart of an ion Bernstein branch. Thus it is of interest to examine in detail these extended fluid models in which a full pressure tensor dynamics is included and to evidenciate their features together with their limitations. With this aim, here we restrict our analysis to the case of an ion pressure tensor and investigate the small amplitude limit of the plasma description adopted in Ref.[8]. More specifically we derive the dispersion relation of the linear waves, propagating in the plane perpendicular to a strong magnetic field, described by this model and compare them to the corresponding kinetic results obtained by an appropriate expansion of the solution of the linearised Vlasov-Maxwell system. We call these waves “Magneto-elastic” waves.

The possibility of treating ion Bernstein modes within a fluid formalism, possibly extending the fluid equations to include high order velocity moments of the particle distribution function, may provide a convenient investigation tool of plasma dynamics far from thermodynamic equilibrium: for example it has been shown within a full Vlasov-Maxwell modelling that the excitation of $\omega \sim 4\Omega$ ion Bernstein waves (where Ω is the ion-cyclotron frequency) in the same geometry configuration we consider here induces a spatially asymmetric, non-gyrotropic heating of ions [9, 10]. In a different context ion Bernstein modes are of interest in dedicated magnetically confined plasma experiments on ion-cyclotron heating mechanisms, such as, e.g., the ALINE [11] and SSWICH [12] experiments that aim to shed light on the convective acceleration and heating of particles in the radio-frequency sheets, which develop both next to the tokamak walls because of the radio-frequency discharges and next to antennas for the ion-cyclotron heating. In Ref.[13] it has been experimentally demonstrated that sheared plasma flows which result from inhomogeneous transverse electric fields in a magnetised plasma induce the excitation of waves propagating in the ion-cyclotron frequency range almost perpendicularly to the magnetic field. Conversely, the excitation of Ion-Bernstein waves is known, both experimentally [14] and theoretically (see Ref.[15] and references therein), to induce the generation of sheared ion flows.

Non-gyrotropic particle distribution functions are measured in the solar wind and at the solar wind-magnetosphere interface and correlation with the presence of shear flows has been evidenced. For example, non-gyrotropic distribution functions have been measured within a magnetic flux tube of ions flowing out into the upstream solar wind [16]. Non-gyrotropic electron distributions have been observed within the electron diffusion region of reconnecting magnetic structures in the magnetopause [17, 18] near X - and O -points, where the stream function of the velocity field is locally hyperbolic (see e.g. Ref.[19]).

In the following we consider the propagation of “Magneto-elastic” waves in a homogeneous Double Adiabatic (Chew-Goldberger-Low, or CGL) plasma equilibrium embedded in an externally imposed magnetic field and look for perturbations with a wave-vector perpendicular to the magnetic field. For the sake of simplicity we take cold, massless electrons. We find two branches of perturbations with fluid velocities and electric fields polarised in the plane perpendicular to the magnetic field: a low frequency branch (LFB) corresponding to magnetosonic waves and a high frequency branch (HFB) corresponding to (generalized i.e., non quasi-electrostatic) ion Bernstein modes. We analyse their dispersion relations and compare their long wave-length behaviour to the kinetic counterparts obtained by linearizing the corresponding Vlasov-Maxwell system and by retaining the $n = 0, \pm 1, \pm 2$ ion cyclotron harmonics in the small Larmor-radius limit. For the sake of brevity in the following we will refer to the dispersion relations obtained from such a truncated kinetic Vlasov ion response as the “kinetic” dispersion relations. Qualitative and quantitative differences in the respective dispersion relations are evidenced. For the low frequency branch we discuss a discrepancy present in the literature on the derivation of the Finite Larmor Radius (FLR) corrections in the CGL-FLR fluid limit.

This article begins with the description of the fluid model (Sec.II) and with its linear analysis in the case of 1D perturbations propagating perpendicularly to a background magnetic field (Sec.III). In Sec.IV the internal consistency of the model is discussed and its limitations are identified by direct comparison with the corresponding kinetic dispersion relations. In Sec.VI a summary of the results obtained is presented and the possibility of describing higher ion cyclotron harmonics ($|n| > 1$) within a fluid framework is noted. Detailed calculations are presented in the Appendices: in Appendix A a brief derivation of the two-fluid model equations is given, in Appendix B the limit of the Double Adiabatic closure is described and in Appendix C the derivation of the FLR limit of the magnetosonic branch

is detailed. We recall in this context that the Double Adiabatic closure (see e.g. [34]) describes the low frequency, long wavelength dynamics of a magnetized plasma with different temperatures, and obeying different equations of state, in the direction of the magnetic field and in the plane perpendicular to it. It assumes temperature isotropy in this plane. This closure applies to plasma conditions where heating and cooling processes and/or approximate integrals of motion due to the presence of particle adiabatic invariants affect the parallel and the perpendicular directions differently while it assumes that the smallness of the particle gyroradii and their fast gyration are sufficient to allow for a fluid isotropic description in the perpendicular plane. For the perpendicular propagation considered here only fast magnetosonic waves could propagate if the Double Adiabatic description were applied to the perturbations of the equilibrium.

II. FLUID MODEL WITH FULL PRESSURE TENSOR EVOLUTION

We consider a single-fluid MHD model in which the contributions of the electron pressure and of the electron inertia are neglected and the ion pressure evolves according to the full pressure tensor equation. This model can be obtained from the two-fluid, full pressure tensor equations derived in Appendix A by adding in the standard way the ion and electron momentum equations (Eq.(A3)) and by using the quasi-neutrality assumption $n^e \sim n^i$ for a hydrogen plasma. Thus the magnetic force and the spatial derivatives of the ion pressure tensor are the only forces acting on the plasma (Eq.(2)), while the electron momentum equation appears in the single-fluid system in the form of the Hall-MHD Ohm's law in the induction equation (Eq.(5)). Furthermore we disregard the ion heat flux, $\partial_i Q_{ijk} = 0$ in the ion pressure tensor equation (Eq.(A6)). In the present analysis this simplifying assumption is made as a closure condition whose consistency with the truncated Vlasov model in the low- β limit will be shown later (Sec.IV). Here β is the ratio (suitably defined, see Eq.(17)) between the ion pressure and the magnetic pressure. We note however that, as stressed in Ref.[21], the comparison with the kinetic dispersion relation may be made exact in the context of the low frequency limit of magneto-sonic waves also at $\beta \sim 1$ if the heat-flux contribution is consistently retained (see Sec.V and Appendix C).

Introducing the ion cyclotron frequency Ω and the unit vector components $b_i \equiv B_i/B$, the single-fluid equations read, with standard notation,

$$\partial_t n + \nabla \cdot (n \mathbf{u}) = 0, \quad (1)$$

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = \Omega \frac{\mathbf{J}}{ne} \times \mathbf{b} - \frac{\nabla \cdot \mathbf{\Pi}}{mn}, \quad (2)$$

$$\partial_t \mathbf{\Pi} + \nabla \cdot (\mathbf{u} \mathbf{\Pi}) + \mathbf{\Pi} \cdot \nabla \mathbf{u} + (\mathbf{\Pi} \cdot \nabla \mathbf{u})^T \quad (3)$$

$$- \Omega (\mathbf{\Pi} \times \mathbf{b} + (\mathbf{\Pi} \times \mathbf{b})^T) = 0.$$

The apex “ T ” indicates matrix transpose. Note in passing that Eq.(3) ensures that the pressure tensor remains positive definite over time if it is positive definite at $t = 0$.

In the adopted model the displacement current is neglected in Ampère's law (Eqs.(A7)), as consistent with the quasi-neutrality assumption and the restriction to phase velocities smaller than the speed of light thus leading to the MHD definition of the current density,

$$\mathbf{J} = \frac{c}{4\pi} \nabla \times \mathbf{B}. \quad (4)$$

Then the induction equation coupled with Ohm's law is

$$\partial_t \mathbf{B} = \nabla \times \left\{ \left(\mathbf{u} - \frac{\mathbf{J}}{ne} \right) \times \mathbf{B} \right\}. \quad (5)$$

The third and the fourth terms in Eq.(3) arise from the action on the pressure tensor $\mathbf{\Pi}$ of the spatial derivatives of the velocity field \mathbf{u} which we write in tensor notation as the sum of a strain (that is, of compression and of compressionless shear) and of a vorticity term as

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{3} \left(\frac{\partial u_k}{\partial x_k} \right) \delta_{ij}}_{\text{compression rate}} + \underbrace{\frac{1}{2} \left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \left(\frac{\partial u_k}{\partial x_k} \right) \delta_{ij} \right]}_{\text{compressionless rate of shear}} + \underbrace{\frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right)}_{\text{fluid vorticity}}. \quad (6)$$

strain tensor

III. LINEAR ANALYSIS OF THE FLUID MODEL

We linearise Eqs.(1-5) around a double adiabatic equilibrium $\Pi_{ij}^0 = P_\perp^0 \delta_{ij} + (P_\parallel^0 - P_\perp^0) b_i b_j$ (see App.B) with uniform density n_0 , no velocity ($\mathbf{u}^0 = 0$), uniform magnetic field $\mathbf{B}^0 = B_0 \mathbf{e}_z$. A double adiabatic equilibrium is the most general, spatially uniform configuration compatible with the pressure tensor equation (Eqs.(B1-B2)) in the absence of a shear flow [22]. When the latter is included, a class of spatially inhomogeneous, non-gyrotropic equilibria can be found [23, 24]. Here we consider perturbations with a wave-vector perpendicular to the equilibrium magnetic field, $\mathbf{k} = k \mathbf{e}_x$, since this makes it possible to decouple the dispersion relation of modes with electric and velocity field components perpendicular to the equilibrium magnetic field $B_0 \mathbf{e}_z$ from the modes with parallel components. This simplifies the study of the propagation of perturbations that make the pressure non-gyrotropic in the plane perpendicular to $B_0 \mathbf{e}_z$. Note however that the curl of the Hall term in Eq (5) vanishes in this geometry. Labelling perturbed quantities by a \sim we set

$$\tilde{\mathbf{u}} = \tilde{\mathbf{u}}_\perp e^{i(kx - \omega t)} + \tilde{u}_z \mathbf{e}_z e^{i(kx - \omega t)}. \quad (7)$$

It is convenient to introduce the characteristic Alfvén velocity (c_a) and the ion thermal velocity (v_{th}) as

$$c_a^2 \equiv \frac{B_0^2}{4\pi n_0 m}, \quad v_{th}^2 \equiv 2 \frac{k_B T}{m} = 2 \frac{P_\perp^0}{n_0 m}. \quad (8)$$

We then introduce the ion skin depth (d_i) and the thermal ion Larmor radius (ρ_i),

$$d_i^2 \equiv \frac{c_a^2}{\Omega^2}, \quad \rho_i^2 \equiv \frac{v_{th}^2}{\Omega^2}. \quad (9)$$

In this model in which only the ion temperature contributes to the plasma pressure, the scale separation between d_i and ρ_i is measured by the magnitude of the β parameter,

$$\beta \equiv \frac{v_{th}^2}{c_a^2} = \frac{\rho_i^2}{d_i^2}. \quad (10)$$

Linearising Eq.(2) in this geometry we find the equations for the in-plane velocity components,

$$\tilde{u}_x = \left(\frac{k}{\omega}\right)^2 c_a^2 \tilde{u}_x + \frac{k}{\omega} \frac{\tilde{\Pi}_{xx}}{n_0 m}, \quad (11)$$

$$\tilde{u}_y = \frac{k}{\omega} \frac{\tilde{\Pi}_{xy}}{n_0 m}, \quad (12)$$

which are coupled to the equations for the perpendicular pressure components, as obtained from Eqs.(3),

$$\frac{\omega}{\Omega} \tilde{\Pi}_{xx} = 3P_\perp^0 \frac{k \tilde{u}_x}{\Omega} + 2i \tilde{\Pi}_{xy}, \quad (13)$$

$$\frac{\omega}{\Omega} \tilde{\Pi}_{xy} = P_\perp^0 \frac{k \tilde{u}_y}{\Omega} + i(\tilde{\Pi}_{yy} - \tilde{\Pi}_{xx}), \quad (14)$$

$$\frac{\omega}{\Omega} \tilde{\Pi}_{yy} = P_\perp^0 \frac{k \tilde{u}_x}{\Omega} - 2i \tilde{\Pi}_{xy}, \quad (15)$$

and to the equation for the parallel pressure component. The latter evolves because of the plasma compressibility,

$$\omega \tilde{\Pi}_{zz} = P_\parallel^0 k \tilde{u}_x. \quad (16)$$

After a few algebraic steps the linearised system equations can be cast in the form $[\mathbf{M}] \cdot \tilde{\mathbf{u}} = 0$, with the dispersion matrix given by:

$$[\mathbf{M}] \equiv \begin{pmatrix} 1 - \frac{k^2(c_a^2 + v_{th}^2)}{\omega^2} + \frac{k^2 v_{th}^2}{2(4\Omega^2 - \omega^2)} & i \frac{\Omega}{\omega} \frac{k^2 v_{th}^2}{(4\Omega^2 - \omega^2)} \\ -i \frac{\Omega}{\omega} \frac{k^2 v_{th}^2}{(4\Omega^2 - \omega^2)} & 1 + \frac{k^2 v_{th}^2}{2(4\Omega^2 - \omega^2)} \end{pmatrix}. \quad (17)$$

A further set of closed equations is provided by the parallel component of the velocity, \tilde{u}_z , which is decoupled from the perturbations $(\tilde{u}_x, \tilde{u}_y, 0)$ but is coupled to the linear equations for $\tilde{\Pi}_{xz}$ and $\tilde{\Pi}_{yz}$,

$$\tilde{u}_z = \frac{k}{\omega} \frac{\tilde{\Pi}_{xz}}{n_0 m}, \quad (18)$$

$$\frac{\omega}{\Omega} \tilde{\Pi}_{xz} = P_{\perp}^0 \frac{k \tilde{u}_z}{\Omega} + i \tilde{\Pi}_{yz}, \quad (19)$$

$$\frac{\omega}{\Omega} \tilde{\Pi}_{yz} = -i \tilde{\Pi}_{xz}. \quad (20)$$

This set provides the dispersion relation of a spurious mode which is unphysical having no counterpart in the Vlasov description:

$$\left(1 + \frac{k^2 v_{th}^2}{2(\Omega^2 - \omega^2)}\right) \tilde{u}_z = 0, \quad (21)$$

(the differences with respect to the corresponding branch obtained from a kinetic treatment are evidenced in Sec.IV). In fact this mode would correspond to a hydrodynamic mode with no perturbed electromagnetic fields and thus cannot be realized in a collisionless plasma. This feature originates from the simplified description of the parallel dynamics adopted in the present fluid model which, for $\mathbf{k} \perp \mathbf{B}_0$ and in the limit of cold mass-less electrons, does not make it possible to balance a parallel electric field component but it does not affect the modes with perturbed velocity perpendicular to the equilibrium magnetic field treated in the rest of this paper. We call the modes described by Eqs.(11-16) “*magneto-elastic* modes”, as they arise because of the pressure anisotropy which couples the electromagnetic perturbations described by Eq.(11) to elastic-type perturbations related to the \tilde{u}_y component (Eq.(12)) through the off-diagonal component $\tilde{\Pi}_{xy}$. The evolution of Π_{zz} , though not directly contributing to the dynamics of $\tilde{\mathbf{u}}_{\perp}$, is coupled to it (see Eq.(16)) because of the compression related to the velocity component \tilde{u}_x (Sec.III A).

When comparing the full pressure tensor dispersion relation with the isotropic MHD model where the magnetosonic branch has dispersion relation $\omega^2 = k^2(c_A^2 + c_s^2)$, with c_s sound speed of the plasma, we need to recall that, since the electrons are taken to be cold, here the sound velocity is replaced by v_{th} of Eq.(8). In particular, the factor 2 in the definition of v_{th}^2 corresponds to a polytropic index $\Gamma_{\perp} = 2$, as follows from the double adiabatic equation for P_{\perp} (Eq.B7), when $\mathbf{B}_0 \cdot \mathbf{k} = 0$.

It is convenient to write the dispersion relation of the modes $\tilde{\mathbf{u}} = \tilde{\mathbf{u}}_{\perp}$ as

$$\frac{\left(\frac{\omega^2}{\Omega^2} - 4 - \frac{k^2 \rho_i^2}{2}\right) \left[\frac{\omega^2}{\Omega^2} - k^2 \left(d_i^2 + \frac{3}{2} \rho_i^2\right)\right] - 2k^2 \rho_i^2}{\omega^2 (4\Omega^2 - \omega)} = 0, \quad (22)$$

which describes the two branches

$$\left(\frac{\omega_{h,l}}{\Omega}\right)^2 = 2 + k^2 \left(\frac{d_i^2}{2} + \rho_i^2\right) \pm 2 \sqrt{\left(1 - \frac{k^2}{4} (d_i^2 + \rho_i^2)\right)^2 + \frac{k^2 \rho_i^2}{2}}, \quad (23)$$

where, consistent with the notation of Ref.[8], the higher frequency branch (HFB), corresponding to the root with the plus sign in front of the square root, is denoted by ω_h while the lower frequency branch (LFB), corresponding to the minus sign, is denoted by ω_l . The behaviour of these dispersion relations is exemplified in Fig.1, left frame. For $\omega < 2\Omega$ only the LFB propagates. It corresponds to a fast magnetosonic wave, corrected by the presence of the full pressure tensor evolution. From a local expansion of the matrix $[\mathbf{M}]$ we see that this branch is compressive and has a purely transverse electric field at low frequencies ($\omega/\Omega \ll 1$), with eigenmodes of components

$$(\tilde{u}_x, \tilde{u}_y) = \left(1, i \frac{\Omega}{\omega_l} \frac{k^2 \rho_i^2}{4}\right), \quad (\tilde{E}_x, \tilde{E}_y) = \left(-i \frac{\Omega}{\omega_l} \frac{k^2 \rho_i^2}{4}, 1\right), \quad (24)$$

while it becomes approximately right-hand circularly polarised, $(\tilde{u}_x, \tilde{u}_y) \sim i(\tilde{E}_x, \tilde{E}_y) \simeq (1, i)$ at the crossing of the “resonance” $\omega \simeq 2\Omega$. At $\omega = 2\Omega$ the HFB appears at $k = 0$: its dependence on k is initially flat and grows linearly for larger values of k . This HFB is related to the $n = 2$ “generalised” (i.e. not quasi-electrostatic [25, 26]) ion-Bernstein mode which is found in a kinetic description (see Sec.IV A). For $k d_i \ll 1$ this mode is left-hand circularly polarised

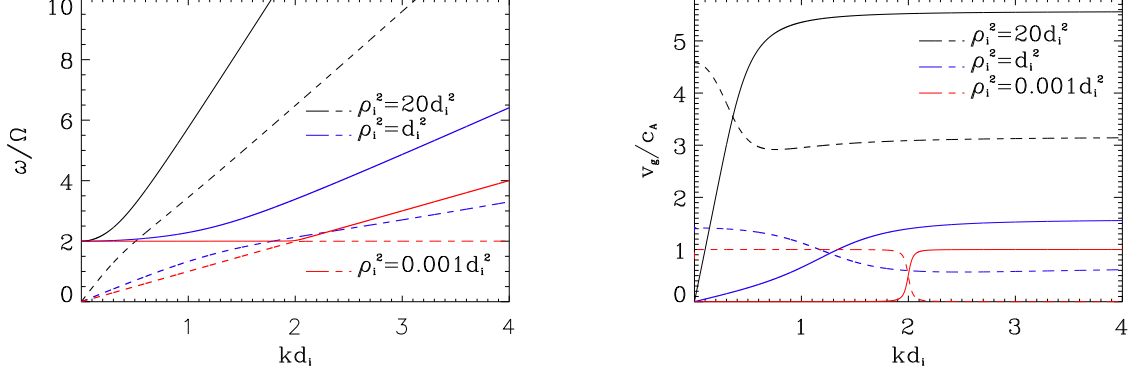


FIG. 1. Dispersion relations (left frame) and group velocities versus kd_i (right frame) of the LFB (ω_l and v_l , dashed lines) and of the HFB (ω_h and v_h , solid lines) obtained from the fluid model for different values of $v_{th}/c_A = \rho_s/d_i$.

with $(\tilde{u}_x, \tilde{u}_y) \sim i(\tilde{E}_x, \tilde{E}_y) \simeq (1, -i)$, and maintains approximately the same polarisation for $kd_i \sim 1$ (as deduced by a local expansion of the eigenmode equations around the resonance). This branch is not present in a double adiabatic description and it arises within a fluid description because the full pressure tensor equations allow for non-gyrotropic pressure perturbations in the x - y plane. The lower frequency bound of this branch, $\omega = 2\Omega$, follows from the fact that, because of the equilibrium magnetic field, these components rotate in the x - y plane at twice the cyclotron frequency. The group velocity of the two branches, normalised to the Alfvén velocity, can be expressed as

$$\frac{v_{h,l}}{c_a} = \frac{kc_a}{\omega_{h,l}} \left\{ \frac{1}{2} + \frac{\rho_i^2}{d_i^2} \pm \frac{A}{B} \right\}, \quad (25)$$

where the \pm sign in front of last term corresponds to the high and to the low branches respectively and

$$A = -1 + k^2 \left(\frac{1}{4} + \frac{\rho_i^2}{2d_i^2} \right) + \frac{k^2 \rho_i^4}{4d_i^2}, \quad (26)$$

$$B = 2\sqrt{\left(1 - \frac{k^2}{4} (d_i^2 + \rho_i^2) \right)^2 + \frac{k^2 \rho_i^2}{2}}. \quad (27)$$

Its behaviour is exemplified in Fig.1, right frame, for some values of d_i and ρ_i .

Note that in the limit $\rho_i \rightarrow 0$ (cold ion limit, vanishing pressure tensor) the two branches come close to crossing each other (see left frame of Fig.1) and that for $\rho_i = 0$ the HFB disappears (in formal terms the solution $\omega^2/\Omega^2 - 4 = 0$ of Eq.(23) is cancelled by the corresponding resonant term in the denominator of Eq.(22)).

Ordering $\rho_i \sim d_i$, in the long wavelength limit ($\rho_i^2 \sim d_i^2$, $k \rightarrow 0$) we obtain

$$\omega_l^2 \simeq k^2(c_a^2 + v_{th}^2), \quad \omega_h^2 \simeq 4\Omega^2 + k^2 v_{th}^2, \quad (28)$$

whereas in the opposite case of very short wavelengths ($\rho_i^2 \sim d_i^2$, $k \rightarrow \infty$ where, however, a fluid-type description is not expected to hold as mentioned at the end of this section) we have

$$\omega_l^2 \simeq \frac{k^2 v_{th}^2}{2}, \quad \omega_h^2 \simeq k^2 \left(c_a^2 + \frac{3}{2} v_{th}^2 \right). \quad (29)$$

Note that in this formal limit the off-diagonal terms of the matrix tend to zero and thus, as expected in the case of short wavelengths, the two branches separate into a purely electrostatic ($\omega_h^2 \simeq k^2[c_a^2 + (3/2)v_{th}^2]$) and a purely electromagnetic ($\omega_l^2 \simeq k^2 v_{th}^2/2$) mode.

For comparison with the CGL-FLR model discussed in Sec.III B it is interesting to consider the first order corrections to the LFB, obtained in the small Larmor radius limit by performing the ordering $k^2 \rho_i^2 \ll k^2 d_i^2$ and $k^2 \rho_i^2 \ll |4 - k^2 d_i^2|$. We find

$$\omega_l^2 \simeq k^2 \left[c_a^2 \left(1 - \frac{k^2 \rho_i^2}{2(4 - k^2 d_i^2)} \right) + v_{th}^2 \left(1 - k^2 \rho_i^2 \frac{4 - 2k^2 d_i^2}{(4 - k^2 d_i^2)^3} \right) \right]. \quad (30)$$

The inequality $k^2 \rho_i^2 \ll |4 - k^2 d_i^2|$ follows from the fact that a small $k^2 \rho_i^2$ expansion is not valid in the vicinity of the near crossing between the two branches mentioned above.

We conclude this Section by noting that, when compared to the corresponding dispersion relations computed from the truncated Vlasov expansion in Sec.IV, the range of validity of the dispersion relations represented in Fig.1 will be restricted to values $k \rho_i \leq 1$. Since $k \rho_i = k d_i \sqrt{\beta}$, this condition becomes more restrictive with increasing values of β when expressed in terms of d_i . Beyond the interval $k d_i \sqrt{\beta} \leq 1$ it is not possible to provide an interpretation of the fluid hierarchy closure in terms of an expansion of the Bessel functions (see Sec.IV-V). Clearly the fluid description is expected to fail for $k d_i \gg 1$, with its range of validity depending on the branch considered.

A. Compressional effects and fluctuations of B_z

Both branches induce fluctuations in \tilde{B}_z and $\tilde{\Pi}_{zz}$. These fluctuations are coupled through the relationship $\tilde{\Pi}_{zz}/\tilde{B}_z = P_{||}^0/B_0$, since linearisation gives

$$\frac{\tilde{\Pi}_{zz}}{P_{||}^0} = \frac{k \tilde{u}_x}{\omega}, \quad \frac{\tilde{B}_z}{B_0} = \frac{k \tilde{u}_x}{\omega}. \quad (31)$$

This polarisation is coherent with an isothermal closure for the parallel “temperature”, $\tilde{\Pi}_{zz}/(\tilde{n}_0 m)$, where $\tilde{n}/n_0 = k \tilde{u}_x/\omega$ is obtained from linearisation of Eq.(1), since for $\mathbf{k} = k_x \mathbf{e}_x$ the double adiabatic equation for $P_{||}$ (Eq.(B6)) corresponds to a polytropic with index $\Gamma_{||} = 1$.

B. Dispersion relation in the limit of a CGL closure with first order FLR

For further comparison we consider the well known CGL fluid model with first order FLR corrections [27–32] recently re-derived in Ref.[23] by starting from a full-pressure tensor model and by assuming $\omega/\Omega \sim k v_{th}/\Omega \sim \varepsilon \ll 1$. By construction this model projects the linear system of Eq.(17) onto the LFB, since it relies on the small frequency approximation $\omega/\Omega \ll 1$ at the basis of the CGL closure. However, as first noticed in Ref.[33] and in Ref.[25], it fails to describe this magnetosonic branch correctly for the purely perpendicular propagation that we are considering here. This fact was further pointed out in Ref.[21] while discussing how the CGL model with first order FLR corrections may lead to an incorrect description of transversely propagating, low-frequency magnetosonic solitons. Here we remark that the failure of the fluid description of fast magnetosonic waves in the CGL-small FLR limit may be understood by noticing that for $\mathbf{B}^0 \cdot \mathbf{k} = 0$ the CGL closure with first order FLR corrections does not take account of the correct polarisation of the magnetosonic waves which requires $\tilde{u}_y/\tilde{u}_x \sim k d_i \ll 1$. For $\mathbf{B}^0 \cdot \mathbf{k} = 0$, indeed, the maximal ordering $\tilde{u}_x/v_{th} \sim \tilde{u}_y/v_{th} \sim \varepsilon^0$, which the CGL-first order FLR model relies on, is inconsistent with the neglect of ε^2 corrections in Eq.(14) because from Eq.(12) it follows that $\tilde{u}_y/v_{th} \sim \varepsilon$. On the other hand, the long wave-length limit, $k^2 d_i^2 \sim \omega^2/\Omega^2 \ll 1$, of Eq.(30),

$$\omega_l^2 \simeq k^2 \left[c_a^2 \left(1 - \frac{k^2 \rho_i^2}{8} \right) + v_{th}^2 \left(1 - \frac{k^2 \rho_i^2}{16} \right) \right], \quad (32)$$

$$\frac{v_l}{c_a} = \frac{k c_a}{\omega_l} \left\{ 1 - \frac{k^2 \rho_i^2}{4} + \frac{\rho_i^2}{d_i^2} \left(1 - \frac{k^2 \rho_i^2}{8} \right) \right\}, \quad (33)$$

is directly obtained from the set of Eqs.(11-15) after ordering $\omega/\Omega \sim k v_{th}/\Omega \sim \tilde{u}_y/v_{th} \sim \varepsilon$ and $\tilde{u}_x/v_{th} \sim \varepsilon^0$. The relevant calculations are detailed in Appendix C, while in Sec.IV A there is the dispersion relation (Eq.(53)) to be compared with the small- β limit of Eq.(32). Here we recall that by linearising the CGL-first order FLR set of equations for perpendicular propagation we would obtain instead

$$\omega_l^2 = k^2 \left[c_a^2 + v_{th}^2 \left(1 + \frac{k^2 \rho_i^2}{16} \right) \right], \quad (34)$$

with group velocity

$$\frac{v_l}{c_a} = \frac{k c_a}{\omega_l} \left\{ 1 + \frac{\rho_i^2}{d_i^2} \left(1 + \frac{k^2 \rho_i^2}{8} \right) \right\}. \quad (35)$$

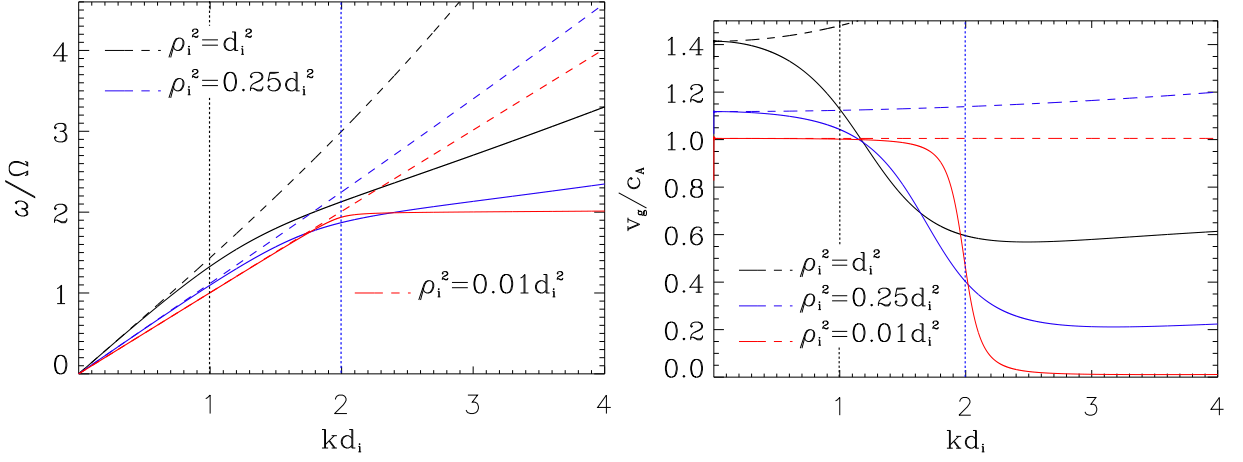


FIG. 2. Dispersion relations (left frame) and group velocities versus kd_i (right frame) for the LFB as obtained by retaining small FLR corrections to the CGL closure for different values of $v_{th}/c_A = \rho_i/d_i = \sqrt{\beta}$: the frequency ω_l from Eq.(34) and the group velocity v_l from Eq.(35) are represented by dashed lines. For comparison with the full pressure tensor model, the curves of ω_l and v_l obtained from Eqs.(23,25) are also represented (solid lines) for the same parameters. For the red dashed curves ($\beta = 10^{-2}$) the condition $k\rho_i < 1$, required by the small-FLR expansion, is fulfilled over the whole interval. For the black ($\beta = 1$) and blue ($\beta = 1/4$) dashed curves the formal ranges of validity $kd_i < 1$ and $kd_i < 2$ are delimited by the black and blue dotted vertical lines, respectively. Even if the above intervals also define the range in which curves with equal colors should be compared, no formal restriction on the wavelength range is assumed for the solid lines which are computed from the full pressure tensor fluid model without expanding for $k\rho_i < 1$.

This latter dispersion relation, first given in Ref. [30] and obtained by neglecting the leading order *perpendicular* (to \mathbf{B}) heat-flux contribution as for Eqs.(32-33), misses the FLR correction to the Alfvénic contribution of Eq.(32) because of the incorrect polarisation (Appendix C). In addition the FLR correction to the thermal velocity appears with the opposite sign. While the $\sim k^2 v_{th}^2 k^2 \rho_s^2$ term should be neglected in the small- β limit $k^2 \rho_i^2 \ll k^2 d_i^2$, in which Eq.(32) coincides with the kinetic counterpart obtained by consistently neglecting the heat-flux gradient contribution (Sec.IV A), in Eq.(34) it represents the only dispersive effect, which is of the same order as the neglected perpendicular heat-flux term.

In Figs.(2) the dispersion relation for the LFB (left frame) and the dependence of the group velocity on k (right frame) at relatively small values of $k\rho_i$ are compared for the solution obtained in the fluid model with a full pressure tensor equation (Eqs.(23,25), solid lines) and in the CGL-FLR model (Eqs.(34-35), dashed lines) for different values of ρ_i/d_i . The range of values of kd_i in Figs.(2), corresponding to values of ρ_i that extend beyond the limit $k\rho_i \sim 1$, has been chosen in order to highlight the differences between the full pressure tensor model and the CGL-FLR limit. This applies in particular to the cases $\rho_i = 0.5d_i$ (blue curves) and $\rho_i = d_i$ (black curves) in which the small FLR limit is often considered, especially for astrophysical applications (see e.g. the discussion in Ref.[23]). Even if the differences may appear negligible in the wave-length range $kd_i \ll 2$ and $k\rho_i \ll 1$, a very different behaviour is displayed for $kd_i \gtrsim 1$, even if $k^2 \rho_i^2 \rightarrow 0$ (red curves). These differences become remarkable for the group velocities, as the small FLR limit obtained from Eq.(30) gives a group velocity, see Eq.(33), that decreases with increasing values of kd_i , a behaviour in agreement with the kinetic result [21, 25, 33] (cf. Eq.(53) next) and in contrast with the increase described by Eq.(35).

C. Eigenmodes in the long wave-length limit

In the long wavelength limit $kd_i \ll 1$ the eigenvectors $\mathbf{V}^l(k, \omega_l)$ and $\mathbf{V}^h(k, \omega_h)$ in the perturbation space $\{\tilde{u}_x, \tilde{u}_y, \tilde{\Pi}_{xy}v_{th}/P_\perp^0, (\tilde{\Pi}_{xx} - \tilde{\Pi}_{yy})v_{th}/P_\perp^0\}$ of the LFB and of the HFB are respectively

$$\mathbf{V}^l = \tilde{u}_x^l \left\{ 1, i \frac{\Omega}{\omega_l} \frac{k^2 \rho_i^2}{4}, \frac{\Omega}{kv_{th}} \frac{k^2 \rho_i^2}{4}, - \frac{k^2 \rho^2}{4\omega_l} \left(\frac{2\omega_l^2 - k^2 v_{th}^2}{kv_{th}} \right) \right\} \quad (\text{LFB}), \quad (36)$$

$$\mathbf{V}^h = \tilde{u}_x^h \left\{ 1, -i, -iv_{th} \frac{\omega_h}{k}, \frac{1}{\Omega} \left(\frac{2\omega_h^2 - k^2 v_{th}^2}{kv_{th}} \right) \right\} \quad (\text{HFB}). \quad (37)$$

where the relative sign between the pressure perturbations and the velocity perturbations depends on the propagation direction i.e., on the sign of the phase velocity. Note that, because of the presence of the two branches and of their different propagation properties, an initial incompressible perturbation of the form $\mathbf{V} = \{0, \tilde{u}_0(x), 0, 0\}$ can generate compressible flows since the difference in the propagation velocities removes the cancellation that makes $\tilde{u}_x(x) = 0$ at the initial time. This feature is central to the mechanism of shear-induced agyrotropy discussed in Ref.[8] and cannot occur in the CGL-FLR limit where only the ω_i mode is present.

IV. KINETIC DESCRIPTION OF THE LOW AND OF THE HIGH FREQUENCY BRANCHES

While the low frequency magnetosonic branch has its counterpart in the MHD plasma description, this is not the case for the high frequency branch which cannot be described by the MHD equations. In order to characterise the origin and nature of this latter propagation branch more precisely, in this section we consider for both branches the corresponding kinetic dispersion relations as obtained from the solution of the linearised Vlasov-Maxwell equations. We assume again purely perpendicular propagation.

First we note that in order to account for the formation of a non-gyrotropic pressure tensor in the x - y plane (i.e., in the plane perpendicular to the uniform magnetic field $\mathbf{B}^0 = B_0 \mathbf{e}_z$), in the solution of the linearised Vlasov equation we need to include the $m = \pm 2$ terms in the expansion of the perturbed ion distribution function into harmonics of the gyration angle α in phase space. The combination of the contributions of these terms to the kinetic dispersion relation leads to the $\omega^2 - 4\Omega^2$ resonances that we have encountered in the fluid treatment. For a similar reason, having neglected in the fluid analysis the ion heat flux $\partial_i Q_{ijk}$, we need not include the $m = \pm 3$ (and higher order) terms in this kinetic treatment.

Thus we compute the ion contribution to the permittivity tensor K_{ij} by retaining the $n = 0$, $n = \pm 1$ and $n = \pm 2$ terms in the summation over the cyclotron harmonics (see e.g. Ref.[34], p.404-405). In addition, in order to make the comparison with the fluid treatment meaningful, we take the small Larmor-radius limit by retaining contributions up to the power $\sim k^4 d_i^2 \rho_i^2$ while neglecting those in $\sim k^4 \rho_i^4$. This means that in the elements of the permittivity tensor we have kept up to the linear contributions in $(k\rho_i)^2/2$ and that the comparison with the fluid derivation is restricted to values $(k^2 \rho_i^2)/(k^2 d_i^2) = \beta \ll 1$. Clearly this expansion procedure does not allow us to recover in the low frequency limit the FLR correction to the $k^2 v_{th}^2$ in Eq.(32). On the other hand, it elucidates the inconsistency of the dispersive effects in the CGL-FLR dispersion relation Eq.(34). These specific points will be considered later in this Section.

We compute the electron contribution to the permittivity tensor taking the limits $\omega^2/\Omega_e^2 \rightarrow 0$ and $k^2 \rho_e^2 \rightarrow 0$, with $\Omega_e = eB_0/(m_e c)$ and ρ_e the electron cyclotron frequency and thermal Larmor radius respectively, as consistent with the zero mass (cold) limit assumed in the fluid treatment. Thus we retain only the off-diagonal contributions to the permittivity tensor of the $\mathbf{E} \times \mathbf{B}$ -drift which are given by the low frequency limit ($\omega^2/\Omega_e^2 \ll 1$) of the $n = \pm 1$ terms in the summation over the electron cyclotron harmonics. By adding the ion and the electron contributions in the limits described above we obtain

$$K_{11} = 1 - \underbrace{\omega_{pi}^2 \frac{1 - k^2 \rho_i^2/2}{\omega^2 - \Omega_i^2}}_{n=\pm 1} - \underbrace{\omega_{pi}^2 \frac{k^2 \rho_i^2}{2(\omega^2 - 4\Omega_i^2)}}_{n=\pm 2}, \quad (38)$$

$$K_{12} = -K_{21} = \underbrace{-i \frac{\omega_{pe}^2}{\omega \Omega_e}}_{n=\pm 1 \ (\alpha=e)} - \underbrace{i \frac{\omega_{pi}^2}{\omega} \Omega_i \frac{1 - k^2 \rho_i^2}{\omega^2 - \Omega_i^2}}_{n=\pm 1} - \underbrace{i \frac{\omega_{pi}^2}{\omega} \Omega_i \frac{k^2 \rho_i^2}{\omega^2 - 4\Omega_i^2}}_{n=\pm 2}, \quad (39)$$

$$K_{22} = 1 - \underbrace{\frac{k^2 v_{th}^2}{\Omega^2} \frac{\omega_{pi}^2}{\omega^2}}_{n=0} - \underbrace{\omega_{pi}^2 \frac{1 - 3k^2 \rho_i^2/2}{\omega^2 - \Omega_i^2}}_{n=\pm 1} - \underbrace{\omega_{pi}^2 \frac{k^2 \rho_i^2}{2(\omega^2 - 4\Omega_i^2)}}_{n=\pm 2}, \quad (40)$$

$$K_{33} = 1 - \underbrace{\left(1 - \frac{k^2 v_{th}^2}{2\Omega^2}\right) \frac{\omega_{pi}^2}{\omega^2}}_{n=0} - \underbrace{\omega_{pi}^2 \frac{k^2 \rho_i^2}{2(\omega^2 - \Omega_i^2)}}_{n=\pm 1}. \quad (41)$$

where $\omega_{pe}^2/\Omega_e = \omega_{pi}^2/\Omega_i$. The order n of the cyclotron harmonic from which each term comes is labelled in the underbrace. Note that the cancellation between the electron and the ion $\mathbf{E} \times \mathbf{B}$ contributions to the plasma current leads in the low frequency limit to a partial cancellation between the first two terms in K_{12} , and K_{21} .

Neglecting the displacement current, as consistent with the fluid model used above, i.e., by taking $c_a^2 \ll c^2$, we write the normalised dielectric tensor $D_{ij} = (\Omega^2/\omega_{pi}^2)\{K_{ij} - (k^2\delta_{ij} - k_i k_j)c^2/\omega^2\}$ as

$$[\mathbf{D}] \equiv \begin{pmatrix} \frac{\Omega^2 - k^2 v_{th}^2/2}{\Omega^2 - \omega^2} + \frac{k^2 v_{th}^2}{2(4\Omega^2 - \omega^2)} & i\frac{\Omega}{\omega} \left(\frac{\omega^2 - k^2 v_{th}^2}{\Omega^2 - \omega^2} + \frac{k^2 v_{th}^2}{4\Omega^2 - \omega^2} \right) & 0 \\ -i\frac{\Omega}{\omega} \left(\frac{\omega^2 - k^2 v_{th}^2}{\Omega^2 - \omega^2} + \frac{k^2 v_{th}^2}{4\Omega^2 - \omega^2} \right) & -\frac{k^2(c_a^2 + v_{th}^2)}{\omega^2} + \frac{\Omega^2 - 3k^2 v_{th}^2/2}{\Omega^2 - \omega^2} + \frac{k^2 v_{th}^2}{2(4\Omega^2 - \omega^2)} & 0 \\ 0 & 0 & D_{33} \end{pmatrix}. \quad (42)$$

The vanishing of the coefficient

$$D_{33} \equiv \frac{k^2 v_{th}^2}{2(\Omega^2 - \omega^2)} - \frac{\Omega^2 + k^2(c_a^2 - v_{th}^2/2)}{\omega^2} \quad (43)$$

yields the dispersion relation of the so-called Ordinary mode,

$$\omega^2 = \Omega^2 - \frac{k^2 v_{th}^2}{2(1 + k^2 d_i^2)}, \quad (44)$$

which corresponds to electromagnetic perturbations with $\tilde{\mathbf{E}}$ parallel to \mathbf{B}_0 . This mode has no counterpart in the fluid description in Sec. III where, as already mentioned, the “hydrodynamic” root related to \tilde{u}_z in Eq.(17) does not apply to a collisionless plasma.

The dispersion relation of the transversely polarised modes, obtained from the vanishing of the determinant of Eq.(42) by consistently neglecting the $\sim k^4 \rho_i^4$ contributions while keeping the $\sim k^4 d_i^2 \rho_i^2$ terms, is

$$\frac{\left(4 - \frac{\omega^2}{\Omega^2}\right) \left[\frac{\omega^2}{\Omega^2} - k^2 d_i^2 \left(1 - \frac{k^2 \rho_i^2}{2}\right) \right] - \frac{k^2 \rho_i^2}{2} \left[k^2 d_i^2 \left(1 - \frac{\omega^2}{\Omega^2}\right) + 8 \right]}{\omega^2(\Omega^2 - \omega^2)(4\Omega^2 - \omega^2)} = 0. \quad (45)$$

Its roots, whose behaviour versus $k d_i$ is sketched in the left frame of Fig.3, are (same notation as in Eq.(23))

$$\left(\frac{\omega_{h,l}}{\Omega}\right)^2 = 2 + \frac{k^2 d_i^2}{2} \pm \sqrt{4 - 2k^2 d_i^2 - 4k^2 \rho_i^2 + \frac{k^4 d_i^2}{4} (d_i^2 + 6\rho_i^2)}. \quad (46)$$

The group velocities, represented in Fig.(3), right frame, are given by

$$\frac{v_{h,l}}{c_a} = \frac{k c_a}{\omega_{h,l}} \left\{ \frac{1}{2} \pm \frac{A}{B} \right\}, \quad (47)$$

where

$$A = -1 - 2\frac{\rho_i^2}{d_i^2} + \frac{k^2 d_i^2}{4} \left(1 + 6\frac{\rho_i^2}{d_i^2}\right), \quad (48)$$

$$B = \sqrt{4 - 2k^2 d_i^2 - 4k^2 \rho_i^2 + \frac{k^2 d_i^2}{4} (k^2 d_i^2 + 6k^2 \rho_i^2)}. \quad (49)$$

As in the fluid description, see Eqs.(22-27), in the limit $\rho_i = 0$ the HFB vanishes as indicated by the cancellation of the $4\Omega^2 - \omega^2 = 0$ terms in the numerator and in the denominator of Eq.(45).

The LFB and HFB described by the roots of Eq.(46) are respectively the magneto-acoustic mode and a generalized ion-Bernstein mode. In the quasi-neutral limit considered here this mode is not quasi-electrostatic, as consistent with the general dispersion relation discussed in Refs.[25], [26].

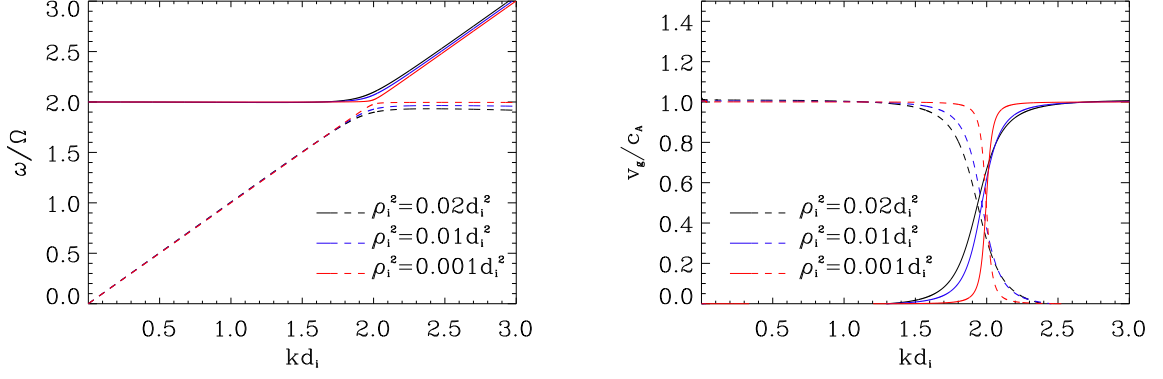


FIG. 3. Dispersion relations (left frame) and group velocities versus kd_i (right frame) of the LFB (ω_l and v_l , dashed lines) and of the HFB (ω_h and v_h , solid lines) obtained from the truncated Vlasov-Maxwell system for different values of $v_{th}/c_A = \rho_i/d_i = \sqrt{\beta} < 1$.

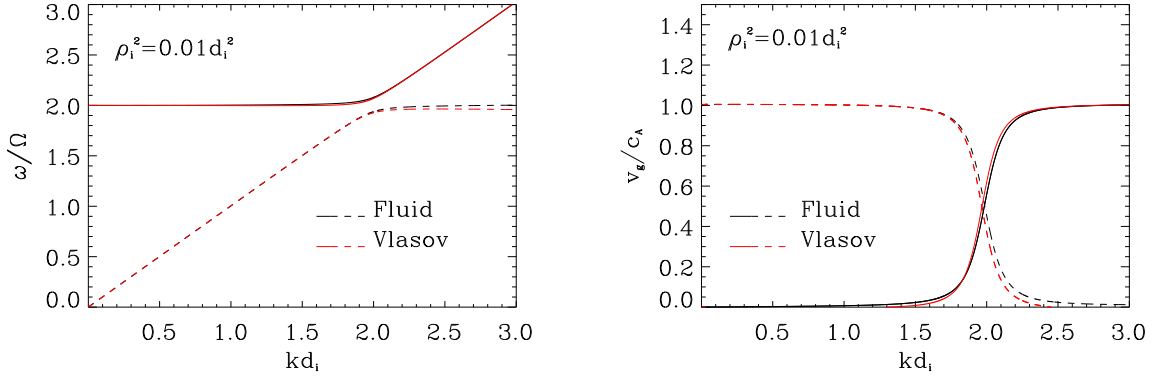


FIG. 4. Dispersion relations (left frame) and group velocities versus kd_i (right frame) of the HFB ω_h obtained from the truncated Vlasov-Maxwell system (red lines) and from the fluid model (black lines). Both the HFB (solid lines) and the LFB (dashed lines) are represented. The value $v_{th}/c_A = \rho_i/d_i = \sqrt{\beta} = 0.01$ has been chosen as representative for the $k^2 \rho_i^2 \ll 1$ limit in the interval of wave-numbers considered.

A. Comparison of the fluid and kinetic dispersion relations

Because of the assumptions made in the derivation of the kinetic dispersion relations (Eqs.(45-49)), a meaningful comparison with the fluid counterparts (Eqs.(22-27)) is possible only for small values of $k^2 \rho_i^2$ i.e., for relatively small values of the ratio $\rho_i/d_i = \sqrt{\beta}$, and for a restricted range of values of kd_i . This comparison is shown in Fig.(4).

First we note that in this small $k^2 \rho_i^2$ limit the HFB described by Eq.(46) has, near the resonance $\omega = 2\Omega$, the factorized expression

$$\frac{\delta\omega}{\Omega} \sim -\frac{\rho_i^2}{d_i^2} f(k^2 d_i^2) = -\frac{\rho_i^2}{d_i^2} \frac{(k^2 d_i^2)(3k^2 d_i^2/2 - 4)}{2(4 - k^2 d_i^2)}. \quad (50)$$

We see that the dependence of $\delta\omega \equiv \omega - 2\Omega$ on ρ_i^2 appears only through a multiplicative factor whereas the values of kd_i corresponding to its minimum (zero of the group velocity) and to the second crossing of the resonance at $kd_i = 2\sqrt{2/3}$, are independent of ρ_i/d_i . On the contrary in the fluid model the HFB remains above the $\omega = 2\Omega$ resonance for all $k \neq 0$. This different behaviour is made evident by the opposite spatial dispersion of the HFB for $k^2 \rightarrow 0$ in the two models where we find

$$\omega_h^2 \simeq 4\Omega^2 \pm k^2 v_{th}^2, \quad v_h \simeq \pm \frac{k v_{th}^2}{2\Omega}, \quad (51)$$

where the plus sign refers to the fluid model and the minus sign to the kinetic treatment. This opposite behaviour is also shown by the first order correction to the ratio of the perturbed pressure tensor components,

$$\tilde{\Pi}_{xx}/\tilde{\Pi}_{yy} \simeq -(1 \pm k^2 \rho_i^2), \quad (52)$$

where again the plus sign refers to the fluid model. The perturbed pressure components are computed in the fluid model directly from the linearised pressure tensor equations and in the kinetic treatment from the second order velocity moment of the perturbed distribution function (Ref.[34], Eq.8.10.8).

Secondly we note that while in the fluid model the LFB magnetosonic branch crosses the $\omega = 2\Omega$ resonance at a value which in the limit $k^4 \rho_i^4 \ll 1$ is $kd_i \simeq 2\sqrt{2}$ (as obtained from Eq.(23)), in the kinetic treatment this branch stays below this resonance. More importantly, having retained only the $k^2 \rho_i^2$ corrections in both the diagonal and off-diagonal matrix elements of the dielectric tensor, the low frequency limit of the kinetic dispersion relation (Eq.(46)) can account consistently only for the FLR corrections to c_a^2 in the dispersion relation of the magnetosonic branch.

The magnetosonic dispersion relation obtained by disregarding the $\sim k^4 \rho_i^4$ terms in the expansion of the LFB of Eq.(46) is

$$\omega_l^2 \simeq k^2 \left[c_a^2 \left(1 - \frac{k^2 \rho_i^2}{8} \right) + v_{th}^2 \right]. \quad (53)$$

This equation agrees with the small- β limit of the kinetic result obtained in Refs. [21, 25] and coincides with the consistent limit of Eq.(32).

V. ROLE OF THE HEAT-FLUX IN THE LONG WAVE-LENGTH LIMIT OF THE LFB FOR $\beta \sim 1$

Neglecting $\sim k^4 \rho_i^4$ contributions in the power expansion of Eq.(46) allows us to obtain the same LFB dispersion relation (53) from the truncated Vlasov-Maxwell system and from the fluid model, closed so as to include the full pressure tensor equation without the divergence of the heat flux. This however does not imply that the heat-flux can always be consistently neglected even when only terms that scale with the perpendicular wave number as k^4 are included.

For β of order unity and for purely perpendicular propagation, terms of order $\sim k^2 v_{th}^2 k^2 \rho_i^2$ must be retained in the dispersion relation of the LFB in the long wave-length limit. In the fluid description a term of this order arises from the heat flux in the limit $\omega \ll \Omega$ and needs to be retained for a consistent description of the LFB, see Appendix C. This feature of the LFB was first pointed out and extensively discussed in Ref.[21]. In the kinetic derivation a term of the same order arises from the $n = 0$ contribution to the K_{22} element of the permittivity tensor which, for $\omega/\Omega \ll 1$, requires us to retain an additional $\sim k^2 \rho_i^2$ term in the expansion of the Bessel function since this term contributes to the dispersion relation of the LFB to the same order of accuracy, in powers of $(\omega/\Omega)^2 \sim k^2 \rho_i^2$, as the other contributions to Eq.(40). In doing so Eq.(40) becomes

$$K_{22} = 1 - \underbrace{\frac{k^2 v_{th}^2}{\Omega^2} \left(1 - \frac{3}{4} k^2 \rho_i^2 \right) \frac{\omega_{pi}^2}{\omega^2}}_{n=0} - \underbrace{\omega_{pi}^2 \frac{1 - 3k^2 \rho_i^2/2}{\omega^2 - \Omega_i^2}}_{n=\pm 1} - \underbrace{\omega_{pi}^2 \frac{k^2 \rho_i^2}{2(\omega^2 - 4\Omega_i^2)}}_{n=\pm 2} \quad (54)$$

and the D_{22} element of matrix (42) is rewritten as

$$D_{22} = - \frac{k^2 \left[c_a^2 + v_{th}^2 \left(1 - \frac{3}{4} k^2 \rho_i^2 \right) \right]}{\omega^2} + \frac{\Omega^2 - 3k^2 v_{th}^2/2}{\Omega^2 - \omega^2} + \frac{k^2 v_{th}^2}{2(4\Omega^2 - \omega^2)}. \quad (55)$$

This leads to the correct extension of Eq.(53) including $\sim k^2 v_{th}^2 k^2 \rho_i^4$ terms, which reads [21, 25]

$$\omega_l^2 \simeq k^2 \left[c_a^2 \left(1 - \frac{k^2 \rho_i^2}{8} \right) + v_{th}^2 \left(1 - \frac{5}{16} k^2 \rho_i^2 \right) \right]. \quad (56)$$

Finally we note that the $|n| = 1$ harmonics are absent from the dispersion relation of both the fluid and the “truncated” Vlasov analysis. In order to recover the $|n| = 1$ harmonics in the dispersion relation obtained from the truncated Vlasov derivation terms proportional to $(k_\perp \rho_i)^4$ must be retained and the limit to $\omega \rightarrow \Omega$ and $(k_\perp \rho_i)^2 \rightarrow 0$ must be performed in an appropriate order. Correspondingly, it can be shown that the $|n| = 1$ harmonics are recovered within the fluid description if the divergence of the heat flux tensor is retained in the equation for the time evolution of the pressure tensor. Consistently with the truncated Vlasov formulation this contribution is proportional to $(k_\perp \rho_i)^4$.

VI. CONCLUSION

The aim of this article is to characterise the linear modes that can propagate in a homogeneous plasma in the direction perpendicular to an externally imposed magnetic field within a two-fluid description that uses a simplified model for the cold electron fluid and a full pressure tensor treatment of the ion fluid. The perturbed velocity fields of these modes lie in the plane perpendicular to the magnetic field. This model description in its nonlinear version has been used in the investigation of the generation of a non gyrotropic pressure tensor [8]. It is thus important to have an *a priori* understanding of the physical properties of this model and of its limitations, in particular since it has been examined in full detail only in the context of its equilibrium configurations [24]. A first step in this line is to determine which kind of waves this model describes and in particular how well they correspond to the results obtained by solving the Vlasov equation. Such a comparison is more easily performed under the homogeneous plasma conditions adopted in the previous sections.

More specifically we have compared the linear dispersion relations derived from this fluid model with those obtained from the solution of the Vlasov equation in a magnetised plasma by retaining, in the appropriate expansion in the thermal ion Larmor radius, the contribution of the $n = 0, \pm 1, \pm 2$ ion cyclotron harmonics. The latter harmonics are required in order to account for the dynamics of an anisotropic, and in general non gyrotropic, pressure tensor. This comparison shows that plasma kinetic dynamics involving the second cyclotron resonance can be mimicked, in the appropriate small Larmor radius limit, by a fluid description provided a full tensor pressure dynamics is retained. Even if the agreement between the two descriptions is not exact in some cases, as explicitly mentioned in our analysis, the overall framework is coherent enough to make the extended fluid model viable and useful when spatially extended inhomogeneous configurations are considered. In such a case fully kinetic simulations are exceedingly expensive in computational terms.

Besides, this fluid approach makes it possible to identify, more easily than with a full kinetic description, some features of the normal modes propagating in the system, such as the way the dynamical plasma response to perturbations determines the polarisation of these normal modes.

The inclusion of ion thermal fluxes in the plane perpendicular to the magnetic field would extend this description making it possible to recover the $n = \pm 1$ cyclotron harmonics that do not appear in the dispersion relations derived above and to include in principle higher harmonics together with higher order terms in Bessel function expansion. In addition, in Sec. V we have shown that, while corrections arising from higher order terms in Bessel expansion of the $n = 0$ cyclotron harmonic appear to be negligible at high frequencies $\omega \gtrsim \Omega$ for small k , they become important when considering FLR-corrections to the magnetosonic dispersion relation at non-negligible values of β as evidenced in [21, 25] in terms of the contribution of the perpendicular heat flux. In this context of comparison between fluid and kinetic frameworks, we recall that the dispersion relations of magnetised plasma modes such as kinetic magnetosonic modes and, using a different approach, Bernstein waves, have been reconstructed from a fluid-like analysis in Refs. [35, 36].

More specifically we have shown that the fluid model describes two wave branches that involve velocity perturbations perpendicular to the magnetic field: a low frequency (LFB) and a high frequency (HFB) branch with dispersion relations $\omega_l^2 \simeq k^2(c_a^2 + v_{th}^2)$ and $\omega_h^2 \simeq 4\Omega^2 + k^2v_{th}^2$ respectively in the long wavelength limit $kd_i \rightarrow 0$. While the LFB corresponds to the fast magnetosonic waves propagating in a cold-electron plasma, where the thermal ion speed replaces the usual sound-velocity, the HFB finds no fluid counterpart in the isotropic pressure limit. The origin of the HFB can be understood by comparison with the kinetic dispersion relation in the small FLR limit (Sec.IV), from which we recognise that the HFB corresponds to a generalization of the 2Ω ion-Bernstein mode.

Finally we underline that the so called CGL-FLR fluid limit, obtained by expanding to the lowest order FLR corrections the pressure tensor around a zeroth order double adiabatic pressure with no heat fluxes, misses the $k^2v_{th}^2$ corrections and obtains a coefficient different both in magnitude and sign for the $k^2v_{th}^2k^2\rho_i^2$ correction as already evidenced in Refs. [21, 25] (see also Appendix C). The source of this discrepancy is in the incorrect retaining of the magnetosonic branch polarisation in the CGL-FLR model and in the subsequent neglect of the heat flux term, whose contribution needs to be retained at the same order of accuracy.

Appendix A: Kinetic derivation of the two-fluid model with full pressure tensor dynamics

Let us multiply the Vlasov equation for the particle distribution function $f^\alpha(\mathbf{x}, \mathbf{v}, t)$, with $\alpha = e, i$ denoting electrons and ions, by a function $\chi(\mathbf{x}, \mathbf{v})$. Integration over d^3v , under the proper convergence conditions as $|\mathbf{v}| \rightarrow \infty$, leads to the so-called “phase-space conservation theorem”,

$$\frac{\partial}{\partial t} n^\alpha \langle \chi \rangle^\alpha + \nabla_x \cdot n^\alpha \langle \mathbf{v} \chi \rangle^\alpha - n^\alpha \langle \mathbf{v} \cdot \nabla_x \chi \rangle^\alpha - \frac{n^\alpha}{m^\alpha} \langle \mathbf{F}^\alpha \cdot \nabla_v \chi \rangle^\alpha = 0, \quad (\text{A1})$$

where $\mathbf{F}^\alpha(\mathbf{x}, t) = q^\alpha(\mathbf{E}(\mathbf{x}, t) + \mathbf{v}/c \times \mathbf{B}(\mathbf{x}, t))$ and we have introduced the particle density $n^\alpha(\mathbf{x}, t) = \int f^\alpha(\mathbf{x}, \mathbf{v}, t) d^3v$ and the mean value over the α -species distribution $\langle A(\mathbf{x}, t) \rangle^\alpha = (1/n^\alpha(\mathbf{x}, t)) \int A(\mathbf{x}, \mathbf{v}, t) f^\alpha(\mathbf{x}, \mathbf{v}, t) d^3v$. By introducing the mean velocity $\mathbf{u}^\alpha(\mathbf{x}, t) \equiv \langle \mathbf{v} \rangle^\alpha$, the continuity equation (Eq.(A2)) and the Euler equation (Eq.(A3)) are obtained from the zeroth ($\chi = m^\alpha$) and from the first order moment ($\chi = m^\alpha v_i$) respectively. The evolution (Eq.(A4)) of the pressure tensor $\Pi_{ij}^\alpha(\mathbf{x}, t) = n^\alpha m^\alpha \langle v_i v_j \rangle^\alpha - n^\alpha m^\alpha u_i^\alpha u_j^\alpha$ is obtained from the second anisotropic moment, $\chi_{ij} = m^\alpha v_i v_j$. This equation involves the gradients of the heat flux tensor $Q_{kij}^\alpha \equiv \langle m^\alpha n^\alpha (v_k - u_k^\alpha)(v_i - u_i^\alpha)(v_j - u_j^\alpha) \rangle$. No explicit dependence on the electric field is present since the two terms $n^\alpha/m^\alpha (E_i u_j^\alpha + E_j u_i^\alpha)$ can be re-expressed in terms of the other variables by using the momentum equation. Adopting a Cartesian tensor notation with lower indices for the spatial components, the three moment equations for the species α are:

$$\frac{\partial n^\alpha}{\partial t} + \frac{\partial}{\partial x_i} (n^\alpha u_i^\alpha) = 0, \quad (\text{A2})$$

$$\frac{\partial u_i^\alpha}{\partial t} + u_k^\alpha \frac{\partial u_i^\alpha}{\partial x_k} = \frac{q^\alpha}{m^\alpha c} (cE_i + \varepsilon_{ilm} u_l^\alpha B_m) - \frac{1}{m^\alpha n^\alpha} \frac{\partial \Pi_{ik}^\alpha}{\partial x_k}, \quad (\text{A3})$$

$$\frac{\partial \Pi_{ij}^\alpha}{\partial t} + \frac{\partial Q_{kij}^\alpha}{\partial x_k} + \frac{\partial}{\partial x_k} (u_k^\alpha \Pi_{ij}^\alpha) + \frac{\partial u_i^\alpha}{\partial x_k} \Pi_{kj}^\alpha + \frac{\partial u_j^\alpha}{\partial x_k} \Pi_{ik}^\alpha \quad (\text{A4})$$

$$- \frac{q^\alpha}{m^\alpha c} (\varepsilon_{ilm} \Pi_{jl}^\alpha B_m + \varepsilon_{jlm} \Pi_{il}^\alpha B_m) = 0.$$

In the latter ε_{ilm} is the Levi-Civita symbol in three dimensions. Provided some closure condition for Q_{ijk}^α is given, the system of fluid equations above is closed once it is coupled to the equations for the e.m. fields,

$$\frac{\partial E_i}{\partial x_i} = \frac{1}{4\pi} (n^e q^e + n^i q^i), \quad \frac{\partial B_i}{\partial x_i} = 0, \quad (\text{A5})$$

$$\frac{\partial B_i}{\partial t} = -c \varepsilon_{ijk} \frac{\partial E_k}{\partial x_j}, \quad (\text{A6})$$

$$\varepsilon_{ijk} \frac{\partial B_k}{\partial x_j} = \frac{4\pi}{c} J_i + \frac{1}{c} \frac{\partial E_i}{\partial t}, \quad J_i \equiv n^e q^e u_i^e + n^i q^i u_i^i. \quad (\text{A7})$$

Notice that the set of Eqs.(A2-A7) leads to an energy conservation equation of the form

$$\begin{aligned} & \frac{\partial}{\partial t} \left\{ \sum_\alpha \left[\frac{n^\alpha m^\alpha}{2} (u^\alpha)^2 + \frac{\text{tr}\{\Pi^\alpha\}}{2} \right] + \frac{B^2}{8\pi} + \frac{E^2}{8\pi} \right\} = \\ & = -\nabla \cdot \left\{ \sum_\alpha \left[\mathcal{Q}^\alpha + \mathbf{u}^\alpha \cdot \Pi^\alpha + \mathbf{u}^\alpha \left(\frac{\text{tr}\{\Pi^\alpha\}}{2} + \frac{n^\alpha m^\alpha (u^\alpha)^2}{2} \right) \right] + \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} \right\}, \end{aligned} \quad (\text{A8})$$

where on the r.h.s. the heat flow vector $\mathcal{Q}_i^\alpha \equiv Q_{ijk}^\alpha \delta_{jk}/2$ has been introduced.

Appendix B: Two-fluid double adiabatic CGL closure from the full pressure tensor equation

The CGL closure for the species α is obtained from Eq.(A4), in the limit of a sufficiently strong magnetic field and/or sufficiently weak velocity strain, by performing an expansion in powers of $1/(\tau_H^\alpha \Omega_\alpha) \ll 1$, where $\tau_H^\alpha \equiv |\nabla \mathbf{u}^\alpha|^{-1}$ and $\Omega_\alpha \equiv |q^\alpha| B_0 / (m^\alpha c)$, and by considering time scales long with respect to Ω_α^{-1} such that $\partial/\partial t \sim 1/\tau_H^\alpha$. To leading order we obtain

$$\varepsilon_{ilm} \Pi_{jl}^{0,\alpha} B_m + \varepsilon_{jlm} \Pi_{il}^{0,\alpha} B_m = 0, \quad (\text{B1})$$

which gives

$$\Pi_{ij}^{0,\alpha} = P_{\perp}^{\alpha} \delta_{ij} + (P_{\parallel}^{\alpha} - P_{\perp}^{\alpha}) b_i b_j, \quad (\text{B2})$$

where P_{\parallel}^{α} and P_{\perp}^{α} are the parallel and perpendicular pressures, respectively. By contracting Eq.(A4) with δ_{ij} we obtain

$$\frac{d}{dt^{\alpha}} (\Pi_{ij}^{\alpha} \delta_{ij}) + (\Pi_{ij}^{\alpha} \delta_{ij}) \frac{\partial u_k^{\alpha}}{\partial x_k} + 2 \Pi_{ik}^{\alpha} \frac{\partial u_i^{\alpha}}{\partial x_k} = 0, \quad (\text{B3})$$

where

$$\frac{d}{dt^{\alpha}} \equiv \partial_t + u_k^{\alpha} \partial_k. \quad (\text{B4})$$

Similarly, contracting Eq.(A4) with $b_i b_j$ gives

$$\frac{\partial}{\partial t} (\Pi_{ij}^{\alpha} b_i b_j) + \frac{\partial}{\partial x_k} (u_k^{\alpha} \Pi_{ij}^{\alpha} b_i b_j) - \Pi_{ij}^{\alpha} \frac{\partial}{\partial t} (b_i b_j) \quad (\text{B5})$$

$$+ b_i b_j \left(\Pi_{ik}^{\alpha} \frac{\partial u_j^{\alpha}}{\partial x_k} + \Pi_{kj}^{\alpha} \frac{\partial u_i^{\alpha}}{\partial x_k} \right) = 0.$$

Eqs.(B3-B5) are valid regardless of the relative ordering between τ_H^{α} and Ω_{α}^{-1} since $(\varepsilon_{ilm} \Pi_{jl}^{0,\alpha} B_m + \varepsilon_{jlm} \Pi_{il}^{0,\alpha} B_m)$ disappears after contraction with either δ_{ij} or $b_i b_j$. In the double-adiabatic ordering, substituting the leading order term $\Pi_{ij}^{0,\alpha}$ for Π_{ij}^{α} into Eqs.(B3-B5) allows us to derive an equation for the time evolution of P_{\parallel}^{α} and P_{\perp}^{α} . Noting that $\Pi_{ij}^{(0,\alpha)} \partial(b_i b_j) / \partial t = 0$, since $|\mathbf{b}| = 1$ by definition, we obtain

$$\frac{dP_{\parallel}^{\alpha}}{dt^{\alpha}} + P_{\parallel}^{\alpha} \frac{\partial u_k^{\alpha}}{\partial x_k} = -2P_{\parallel}^{\alpha} b_l b_k \frac{\partial u_l^{\alpha}}{\partial x_k}, \quad (\text{B6})$$

$$\frac{dP_{\perp}^{\alpha}}{dt^{\alpha}} + 2P_{\perp}^{\alpha} \frac{\partial u_k^{\alpha}}{\partial x_k} = P_{\perp}^{\alpha} b_l b_k \frac{\partial u_l^{\alpha}}{\partial x_k}. \quad (\text{B7})$$

Eqs.(B6-B7) represent the most general writing of the double adiabatic equations, as deduced in a two fluid model. These reduce to the usual CGL closure [22] of the single-fluid model when we can assume $|\mathbf{u}^e| \sim |\mathbf{u}^i|$. Eqs.(B6-B7), extended to include 1st-order FLR corrections and the vectorial heat-flux contribution, were recently re-derived in Ref.[23], by ordering $\omega / \Omega_{\alpha} \sim 1 / (\tau_H^{\alpha} \Omega^{\alpha}) \sim \rho_{\alpha} / L_H$ with L_H^{-1} characteristic spatial gradient of the fluid velocity.

Appendix C: Small FLR limit for the magnetosonic waves at perpendicular propagation

We order $\omega / \Omega \sim k v_{th} / \Omega \sim \varepsilon \ll 1$. We see from Eq.(11) and Eq.(13) that the ordering $\tilde{u}_x / v_{th} \sim \tilde{\Pi}_{xx} / (n_0 m v_{th}^2) \sim \varepsilon^0$ and $\tilde{\Pi}_{xy} / (n_0 m v_{th}^2) \sim \varepsilon$ can be consistently assumed while from Eq.(15) we can assume $\tilde{\Pi}_{yy} / (n_0 m v_{th}^2) \sim \varepsilon^0$. However, we see from Eq.(12) that the ordering $\tilde{u}_y / v_{th} \sim \varepsilon$ follows, which is indeed coherent with the LF magnetosonic branch polarisation, for which $\tilde{u}_y / \tilde{u}_x \sim i\varepsilon$ (Eq.(24)). This latter condition implies through Eq.(14) the ordering $(\tilde{\Pi}_{yy} - \tilde{\Pi}_{xx}) / (n_0 m v_{th}^2) \sim \varepsilon^2$. We may now write

$$\tilde{\Pi}_{yy} = \tilde{P}_{\perp}^0 + \tilde{\Pi}_{yy}^{(1)}, \quad \tilde{\Pi}_{xx} = \tilde{P}_{\perp}^0 - \tilde{\Pi}_{yy}^{(1)}, \quad (\text{C1})$$

where the apex “(1)” labels the $\sim \varepsilon^2$ correction. This is to ensure the conservation of the trace of Π_{ij} at any order in ε , as suggested by the sum of Eq.(13) and Eq.(15).

Using Eq.(B7), that gives $\tilde{P}_{\perp}^0 = 2P_{\perp}^0 k \tilde{u}_x / \omega$, from Eq.(13) we obtain (cf. Eq.(A17) of [23])

$$\tilde{\Pi}_{xy} = i \frac{P_{\perp}^0}{2} \frac{k \tilde{u}_x}{\Omega}. \quad (\text{C2})$$

Here, only the $\sim \varepsilon$ contribution has been retained, the other terms being at least of order $\sim \varepsilon^3$. On the contrary the last term in Eq.(14) leads to an $\sim \varepsilon^2$ contribution: combining Eq.(C2) with Eq.(C1) we obtain

$$\tilde{\Pi}_{xx}^{(1)} = -\tilde{\Pi}_{yy}^{(1)} = -i \frac{P_{\perp}^0}{2} \frac{k \tilde{u}_y}{\Omega} - \frac{\omega}{\Omega} \frac{P_{\perp}^0}{4} \frac{k \tilde{u}_x}{\Omega}. \quad (\text{C3})$$

This last equation differs from the usual CGL-FLR contribution because of the last term (see Eq.(A16) of [23] for $\partial_y = 0$). This term is neglected when the maximal ordering $\tilde{\Pi}_{yy} - \tilde{\Pi}_{xx} \sim \varepsilon$ related to the assumption $\tilde{u}_y/v_{th} \sim \tilde{u}_x/v_{th} \sim \varepsilon^0$ is performed in Eq.(14), as is normally done in the CGL-FLR model. Solving the linear system of Eqs.(11-12,14-15) by using Eqs.(C1-C3) leads to the dispersion relation and group velocity of the small FLR limit of the LFB (Eqs.(32-33)), which gives an opposite dispersive correction with respect to the CGL-FLR magnetosonic mode (Eqs.(34-35)), besides having a further contribution linear in $v_{th}^2 \rho_i^2$.

Though Eqs.(32-33) are consistent with the appropriate limit of Eq.(53) when the heat-flux contribution is neglected, this latter assumption in a full kinetic treatment implies the restriction to small values of β . In this regard we recall that it has been pointed out in Ref. [21] how to recover also in the fluid $\beta \sim 1$ regime the correct kinetic small-FLR limit of the magnetosonic dispersion relation.

At $\beta \sim 1$ the $\sim k^4 v_{th}^4 / (\omega^2 \Omega^2)$ contributions to the small frequency dispersion relation should be retained. For consistency we must retain also the $\sim k^4 v_{th}^4 / (\omega^2 \Omega^2)$ contribution to the $n = 0$ harmonic in K_{22} of Eq.(40) arising from the next order term in the expansion of the Bessel function, which leads to Eqs.(54-55) and then to the kinetic dispersion relation of the magnetosonic branch that includes the FLR correction to the sound term, too (Eq.56. The kinetic dispersion relation (56) differs from Eq.(32) because of the factor 5 in the $\sim k^2 v_{th}^2 k^2 \rho_i^2$ term. The discrepancy is due to the neglect of the gradient of the heat flux from Eq.(3), whose lowest order contribution to the linear system was shown in Ref.[21] to enter with a term $\partial_x \tilde{Q}_{xy} \sim \varepsilon^2$ in Eq.(14). This thermal flux contribution, first derived in Ref.[37] and more recently re-discussed in Ref.[38], would correct the r.h.s. term of Eq.(C3) with a further term $-ik \tilde{Q}_{xy} / (2\Omega)$, which modifies Eq.(C3) into

$$\tilde{\Pi}_{xx}^{(1)} = -\tilde{\Pi}_{yy}^{(1)} = -i \frac{P_{\perp}^0}{2} \frac{k \tilde{u}_y}{\Omega} - \frac{\omega}{\Omega} \frac{P_{\perp}^0}{4} \frac{k \tilde{u}_x}{\Omega} \left(1 - 2 k^2 \rho_i^2 \frac{\Omega^2}{\omega^2} \right). \quad (\text{C4})$$

The above discussion is summarised by the long wave-length limit of the dispersion matrix (17) for the magnetosonic branch, as obtained by using both Eqs.(C3-C4),

$$[\mathbf{M}] = \begin{pmatrix} 1 - \frac{k^2}{\omega^2} \left[c_a^2 + v_{th}^2 \left(1 - \frac{k^2 \rho_i^2}{4} \right) + \frac{k^2 \rho_i^2}{8} \right] & ik \rho_i \frac{k v_{th}}{\omega} \\ -ik \rho_i \frac{k v_{th}}{\omega} & 1 \end{pmatrix}. \quad (\text{C5})$$

In the standard CGL-FLR approximation the $k \rho_i$ terms are inconsistently set to zero everywhere but in the off-diagonal components. Including the $k^2 \rho_i^2 / 8$ term in square brackets of the M_{xx} element, which is due to the last r.h.s term of Eq.(C3), makes it possible instead to recover the correct magnetosonic dispersion relation of Eq.(53) in the small- β limit. If we let $k d_i \sim k \rho_i$, i.e. $\beta \sim 1$, the lowest order contribution of the heat-flux gradient should also be retained, and from the further contribution of Eq.(C4) to Eq.(C3) the $-k^2 \rho_i^2 v_{th}^2 / 4$ correction in the M_{xx} element follows. In this case Eq.(56) is recovered.

We conclude by noting that, while the assumption $\tilde{u}_y/v_{th} \sim \varepsilon^0$ is incorrect to describe the propagation of magnetosonic waves in this specific geometry where $\mathbf{B}_0 \cdot \mathbf{k} = 0$, it becomes legitimate when the condition $\mathbf{B}_0 \cdot \mathbf{k} = 0$ is relaxed or when compressionless equilibrium conditions dependent e.g. simply on $\mathbf{u} = (0, u_y(x), 0)$ are searched for, as done in Ref.[23]. In the first case, a contribution proportional to $k_{\parallel} c_a^2$ due to the y -component of the $\Omega \mathbf{J} \times \mathbf{b} / (ne)$ force of Eq.(2) would appear at r.h.s. of Eq.(12), which allows us in principle to order $u_y/v_{th} \sim u_x/v_{th} \sim \varepsilon^0$, at least as long as k_{\parallel}/k_{\perp} is not ordered small (i.e. of order ε itself). In the second case, since $u_y(x)$ only enters in the incompressible non-gyrotropic equilibria devised in Ref.[23], no relative ordering is needed between u_x and u_y (and heat fluxes are null at equilibrium). We finally note that corrections to the inconsistent CGL-FLR description of the propagation of fast magnetosonic waves have also been discussed in the different context of the Landau-fluid models [39–41].

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